Setting $d\phi/d\gamma = \infty$, a maximum range condition for $\phi > 180°$, gives

$$\sin\phi_{\max} - \alpha \sin\gamma_{\max} \cos(\gamma - \phi)_{\max} = 0 \qquad (6a)$$

or

$$\cot\phi_{\max} = \frac{1 + \cot^2\gamma_{\max} - \alpha}{\alpha \cot\gamma_{\max}} \qquad (6b)$$

where $\gamma_{\max}$ is the burnout velocity angle for maximum range. The explicit expression for $\phi_{\max}$ is obtained by combining Eqs. (6) with the basic hit equation (5). Equation (5) is manipulated to form a term that is the right side of Eq. (6b), and $\cot\phi_{\max}$ is substituted:

$$\cot^2\gamma_{\max} = \frac{h^2 \sin^2\phi_{\max}}{R^2(1 - \cos\phi_{\max})^2} \qquad (7)$$

Since $\gamma \leq 90°$ when $\pi \leq \phi \leqslant 2\pi$, $\cot\gamma$ is positive, and

$$\tan\gamma_{\max} = -(R/h)\tan(\phi_{\max}/2) \qquad (\text{for } \phi_{\max} > 180°) \qquad (8)$$

Equation (7) is now combined with the basic hit equation (5):

$$\phi_{\max} = \cos^{-1}\left[\frac{\alpha - (h/R)^2 - 1}{\alpha(1 + h/R) + (h/R)^2 - 1}\right] \qquad (9)$$

Differentiating the basic hit equation (5) to obtain $d\phi/d\alpha$, while holding h and γ constant, gives

$$\frac{d\phi}{d\alpha} = \frac{1 - \cos\phi}{\alpha \sin\phi - \alpha^2 \sin^2\gamma \sin\phi - \alpha^2 \sin^2\gamma \cot\gamma \cos\phi}$$

When one sets $d\phi/d\alpha = \infty$, a maximum range condition for $\phi > 180°$ is obtained:

$$\cot\phi_{\max} = \frac{1 + \cot^2\gamma_{\max} - \alpha}{\alpha\cot\gamma_{\max}} \qquad (10)$$

which is identical with Eqs. (6) and leads to Eq. (9).

IV. Discussion

The maximum range $\phi_{\max}$ and the corresponding launch angle are plotted in Figs. 3 and 4 as functions of α for various values of h/R. It can be shown that for $\alpha < 2$ the orbit is elliptical and therefore closed, for $\alpha = 2$ the orbit, is parabolic, and for $\alpha > 2$ the orbit is hyperbolic. In the practical sense, $\phi_{\max} > 180°$ requires a closed orbit and, since parabolic and hyperbolic orbits are not closed, $\alpha < 2$ is the maximum value of α considered. This, then, limits $\phi_{\max}$ for a given burnout altitude other than $h = 0$.

It appears that the curves in Fig. 3 have a common point of intersection at $\alpha = 1.0$, but the use of Eq. (9) will show that this is not the case. As shown in Fig. 3, the larger ranges beyond $\phi_{\max} = 270°$ for a given α require small burnout altitudes. For the burnout altitudes considered, ranges between 180 and 270° require relatively small increases in α over the $\alpha_{\min}$ value for $\phi_{\max} = 180°$. To provide an idea of the energy required for extreme impact ranges, which the ratio α does not give, the energy per pound of a satellite in a circular orbit at 100-naut-mile alt is assumed to be the energy per pound which a projectile has at a burnout altitude of $0.02R$. The burnout α is then 1.01. Similarly, the satellite energy per pound for circular orbits of 200, 500, and 1000 naut miles, when given to a projectile at a $0.02R$ burnout altitude, result in burnout α's of 1.04, 1.11, and 1.21, respectively.

A circular orbit results when $\alpha = 1$ and $\gamma = 90°$. If $h = 0$, the orbit just touches the round model earth everywhere on the orbit. With $h = 0$, $\gamma = 90°$, and $1 < \alpha < 2$, the orbits are tangent to the earth at the launch point, and $\phi_{\max} = 360°$ as indicated in Fig. 3.

The conversion of $\phi_{\max}$ and $\gamma_{\max}$, which are in inertial coordinates, to $\phi_{\max}$ and $\gamma_{\max}$ in earth coordinates will require a determination of the influence of the earth's rotation. This influence will depend on the launch point latitude and the burnout velocity azimuth.

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Observations on Minor Circle Turns

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The equatorial tangent minor circle turn is studied in detail as a generalization and correction of some earlier work by Jackson. The turn described herein provides convenient closed-form solutions for studying lateral maneuvering. It is demonstrated that the turn can be described as a simple geometric relationship in terms of orbital elements by a single parameter *Q*. Position in the turn is then shown to be a function of *L/D* and the available kinetic energy. The bank-angle schedule to fly the turn results in an easily programmed function of velocity.

Nomenclature

m = mass
g = constant of proportionality between weight and mass
V = velocity
r = radius from center of earth to vehicle
L = lift
D = drag
Q = equatorial minor circle parameter
N = defined by Eq. (25a)
i = orbit inclination
φ = latitude
λ = longitude
ψ = heading angle measured from a latitude line
γ = flight path angle
β = bank angle measured from local vertical
η = $(V/V_c)^2$
α = defined by Eq. (23)

Subscripts

i = initial
f = final
c = circular

A MINOR circle turn requires a vehicle to fly in a minor circle rather than in its natural great circle trajectory and necessitates continuous aerodynamic bank control to maintain the flight path. The minor circle may be defined in several ways: Loh[1] prefers to specify the turn by the colatitude, whereas Shaver[2] specifies the turn by the latitude. The polar circles thus defined describe trajectories along a line of constant latitude and give range in terms of the arc length along the turn. Jackson[3] has additionally studied closed-form solutions of lateral maneuvering which give range in terms of orbital elements. An integration error in

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Jackson's analysis was subsequently corrected.[4] As mentioned by Jackson,[5] his maneuver is a minor circle tangent to the equator. The minor circle turn is a useful analytical tool for re-entry studies. It therefore appears useful to provide some additional insight into this programmed maneuver.

For convenience, the derivation of the programmed maneuver for the equilibrium glide assumptions will be presented first:

$$\begin{aligned} &\gamma \ll 1 && \text{small flight path angle} \\ &V\dot{\gamma}/g \ll 1 && \text{no rapid turns} \qquad (1) \\ &L/D = \text{const} \end{aligned}$$

Under these assumptions, the equations of motion according to the notation of Fig. 1 are

$$L \cos\beta - mg + (mV^2/r) = 0 \qquad (2)$$

$$m\dot{V} + D = 0 \qquad (3)$$

$$mV(\dot{\psi} + \dot{\lambda} \sin\varphi) = L \sin\beta \qquad (4)$$

$$V \cos\psi = r\dot{\lambda} \cos\varphi \qquad (5)$$

$$r\dot{\varphi} = V \sin\psi \qquad (6)$$

$$\dot{h} = V\gamma \qquad (7)$$

Equations (2–6) may be combined to give

$$d\psi = -\frac{L}{D}\frac{\sin\beta}{2}\frac{d\eta}{\eta} + \frac{L}{D}\frac{\cos\psi}{2}\tan\varphi \cos\beta \frac{d\eta}{(1-\eta)} \qquad (8)$$

$$d\varphi = -\frac{L}{D}\frac{\cos\beta}{2}\sin\psi \frac{d\eta}{(1-\eta)} \qquad (9)$$

$$d\lambda = \frac{d\varphi}{\tan\psi \cos\varphi} \qquad (10)$$

where $\eta = (V/V_c)^2$.

Equations (8) and (9) are now combined to eliminate $d\eta$ and relate latitude φ and longitude λ. A parameter Q is defined as

$$Q \triangleq \tan\beta[(1-\eta)/\eta] \qquad (11)$$

For constant Q, integration yields

$$\cos\psi \cos\varphi = -Q \sin\varphi + C \qquad (12)$$

For entry at latitude φ_i from an orbit with inclination i,

$$C = \cos i + Q \sin\varphi_i \qquad (13)$$

A general value of C could be carried through the subsequent analysis. For the initially equatorial orbit, $C = 1$, and this value is used subsequently. Longitudinal range is obtained from Eq. (10) by substituting the deflection angle from Eq. (12) to give

$$\frac{d\lambda}{d\varphi} = \frac{1 - Q \sin\varphi}{\cos\varphi[2Q \sin\varphi - (1 + Q^2) \sin^2\varphi]^{1/2}} \qquad (14)$$

Before Eq. (14) is integrated, certain features of the maneuver may be displayed through the singularities and zeros of Eq. (14). Equation (14) has singularities at

$$\sin\varphi = 0 \qquad 2Q/(1+Q^2) \qquad \pm 1 \qquad (15)$$

Now a latitude (φ_*) is defined such that

$$\sin\varphi_* = 2Q/(1+Q^2) \qquad (16)$$

Equation (16) is unaffected by replacing Q by its reciprocal. A zero of Eq. (14) occurs at $\varphi = \varphi_{**}$, defined by

$$\sin\varphi_{**} = 1/Q \qquad (17)$$

Substituting φ_* and φ_{**} into Eq. (12), the heading angles are

$$\psi_* = 0° \text{ or } 180° \qquad (18)$$

$$\psi_{**} = 90° \qquad (19)$$

For $Q > 1$, the trajectory turns back on itself after reaching a north heading at φ_{**}. The maximum latitude attained is φ_*, where $\psi_* = 180°$. At this maximum latitude, the longitude is back to the initial value of zero. For $Q = 1$, the trajectory passes over the pole with $d\lambda/d\varphi = 0$. For $Q < 1$, the maximum latitude is φ_*, and the heading is due east.

It is easily shown that the singularities of Eq. (14) can be integrated. Hence, the longitudinal range is

$$2\lambda = \sin^{-1}\left\{\frac{Q^2 - Q + 1 + [(Q-1)^2/(\sin\varphi - 1)]}{Q}\right\} + \sin^{-1}\left\{\frac{Q^2 + Q + 1 - [(Q+1)^2/(\sin\varphi + 1)]}{Q}\right\} \qquad (20)$$

This is a generalization and correction of the result given in Ref. 3 and reduces to the special case $Q = 1$.[4] Equation (20) is plotted in Fig. 2 with the maneuver index (Q) as a parameter. Figure 2 is distorted near the pole because of the nonconformal mapping of the sphere on the plane; although Fig. 2 is plotted correctly, it does not display the circular nature of the trajectories.

The maneuver index, Q, is equivalent to the specification of a polar minor circle by the latitude, as in Fig. 1b. The force relationships normal and parallel to a plane passing through the latitude are obtained from Eqs. (2, 4, and 5) by setting $\dot{\psi} = \psi = 0$.
Combining,

$$\tan\beta[(1-\eta)/\eta] = \tan\varphi = Q \qquad (21)$$

It is seen from Eq. (21), for example, that a 45° minor circle turn about the pole is equivalent to a $Q = 1$ minor circle tangent to the equator. By simple trigonometric relationships, this result reduces immediately to the result of Shaver[2] when the equilibrium assumptions are applied to his Eq. (5):

$$\sin\beta = \frac{\eta \tan\varphi}{[(1-\eta)^2 + (\eta \tan\varphi)^2]^{1/2}} \qquad (22)$$

Loh[1] uses the colatitude as the turn parameter and measures his bank angle from the pole direction. Hence,

$$\alpha + \beta = 90° - \varphi \qquad (23)$$

Using the relation (23) and Eq. (21), Eq. (8-7) of Ref. 1 is obtained in terms of latitude:

$$\sin\alpha = \frac{\cos\varphi[1 - (\eta/\cos^2\varphi)]}{\{\sin^2\varphi + \cos^2\varphi[1 - (\eta/\cos^2\varphi)]^2\}^{1/2}} \qquad (24)$$

It is therefore suggested that the maneuver outlined here is more useful than the conventional definition of a minor circle turn, since it gives the maneuver directly in terms of orbital elements rather than as arc length along the turn. Thus far, the trajectory has been computed without regard to the ability of the vehicle to sustain itself along the glide. Position in the trajectory is a function of L/D and the available kinetic energy. To display latitude as a function of L/D and speed, Eqs. (11) and (13) are used to integrate Eq. (9):

$$\sin\varphi = \frac{Q}{1+Q^2}\left[1 - \cos\left(\frac{L}{D}\frac{N}{2}\right)\right] \qquad (25)$$

where

$$N = \ln\left\{\frac{-1 + (1+Q^2)\eta_f + (1+Q^2)^{1/2}\,[(1+Q^2)\eta_f^2 - 2\eta_f + 1]^{1/2}}{-1 + (1+Q^2)\eta_i + (1+Q^2)^{1/2}\,[(1+Q^2)\eta_i^2 - 2\eta_i + 1]^{1/2}}\right\} \tag{25a}$$

Equation (25a) has been computed and tabulated for $Q = 1/(3)^{1/2}$, 1, $(3)^{1/2}$ by Nyland.[6] See also Ref. 1.

Equation (25) is overlaid in Fig. 2 as lines of constant L/D. A particular value of N occurs at the maximum latitude, $\varphi = \varphi_*$, such that

$$N = -2\pi/(L/D)$$

For this case, Fig. 3 displays the L/D required as a function of initial speed. The validity of the assumptions in Eq. (1) begins to break down in the final phase of the trajectory, and, although zero final velocity has been used in this analysis, the effect on range is small (<10%). As seen from Fig. 3, the L/D required to reach the maximum latitude φ_* decreases monotonically as the maneuver initiation speed is increased. It also is noted that the required L/D for a given value of the maneuver index Q is the same as for the reciprocal of Q.

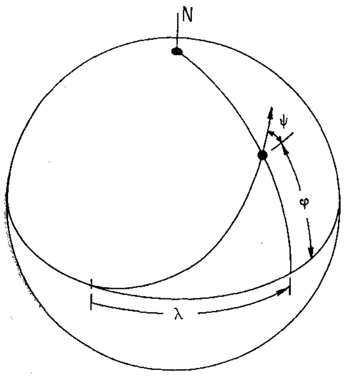

Fig. 1a Equatorial tangent minor circle coordinates.

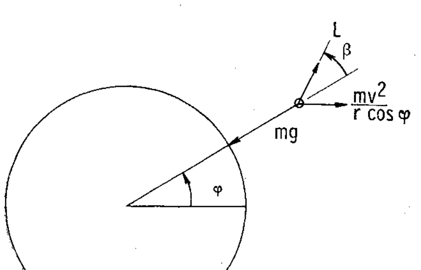

Fig. 1b Polar minor circle force relationships.

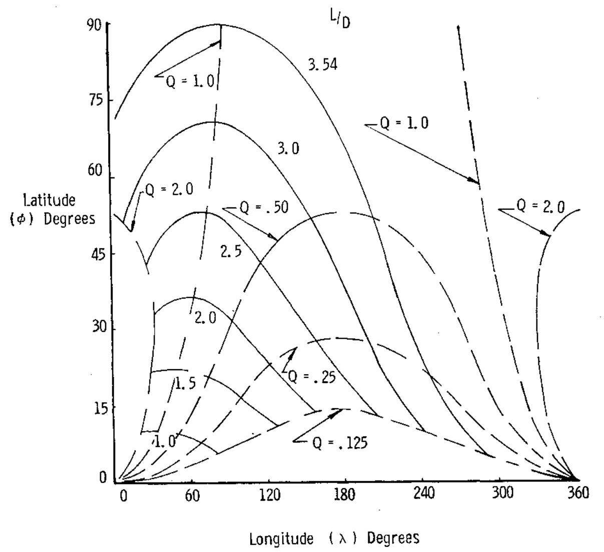

Fig. 2 Equatorial tangent minor circle maneuver for $\eta_i = 1.0$, $\eta_f = 0$.

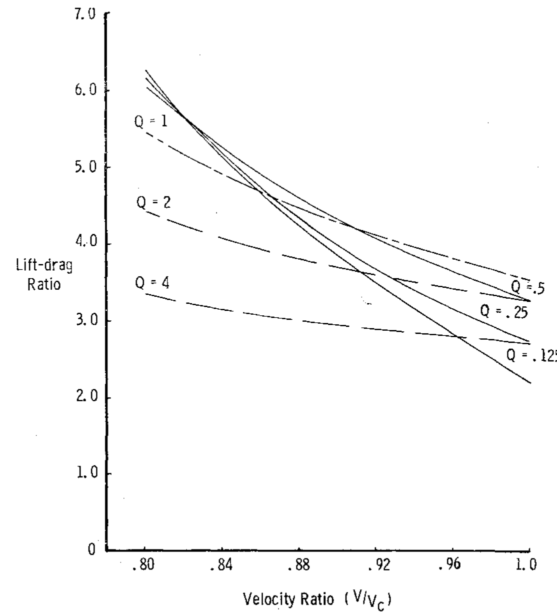

Fig. 3 Required L/D to attain the maximum lateral offset, φ_*, as a function of initial speed.

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Stresses in Solid Propellants Due to High Axial Acceleration

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Nomenclature

G	=	shear modulus
u,v,w	=	displacements in x, y, and z directions, respectively
x,y,z	=	Cartesian coordinate system
z	=	axial coordinate, positive in forward direction of motor
Z	=	body force in z direction
β	=	acceleration loading
γ	=	shear strain
ϵ	=	direct strain
ρ	=	weight density
σ	=	direct stress
τ	=	shear stress
ϕ	=	stress function

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